Transient non-Darcy forced convective heat transfer from a flat plate embedded in a fluid-saturated porous medium

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Transient non-Darcy forced convection on a flat plate embedded in a porous medium is investigated using the Forchheimer-extended Darcy law. A sudden uniform pressure gradient is applied along the flat plate, and at the same time, its wall temperature is suddenly raised to a high temperature. Both the momentum and energy equations are solved by retaining the unsteady terms. An exact velocity solution is obtained and substituted into the energy equation, which then is solved by means of a quasi-similarity transformation. The temperature field can be divided into the one-dimensional transient (downstream) region and the quasi-steady-state (upstream) region. Thus the transient local heat transfer coefficient can be described by connecting the quasi-steady-state solution and the one-dimensional transient solution. The non-Darcy porous inertia works to decrease the velocity level and the time required for reaching the steady-state velocity level. The porous-medium inertia delays covering of the plate by the steady-state thermal boundary layer.

Keywords: porous media; forced convection; transient solution; non-Darcy flow; asymptotic solutions

Introduction

Most of the recent efforts concerning boundary layer flows within porous media have been directed at the problems of steady free or mixed convection over heated bodies, using Darcy's law.[1–4] Transient boundary layer flow problems, however, have received little attention so far. The first study of transient boundary layer flow on a flat plate in a porous medium was by Johnson and Cheng.[5] They found similarity solutions for certain variations of wall temperature. Ingham *et al.*[6] utilized asymptotic expansions to attack the problem of transient free convection on a suddenly cooled vertical flat plate in a porous medium. Cheng and Pop[7] introduced an integral method to analyze the transient free convection boundary layer flow in a porous medium. Cheng and Pop's work was followed by Ingham and Brown,[8] who allowed the wall temperature to vary according to a power function of the distance from the leading edge. Recently, finite difference calculations have become available for the transient forced convection from a heated circular cylinder[9] and transient free convection with mass transfer from a flat plate in porous media.[10]

These studies of transient boundary layer flows were all based on the assumption of Darcy's law, which easily breaks down for fast flows in which porous inertia effects are no longer negligible. Here, we investigate for the first time the transient non-Darcy forced convection from a flat plate embedded in a porous medium by means of a quasi-similarity transformation. The non-Darcy inertia effects are included in the momentum equation through the so-called Forchheimer's extension.[11] Unlike the previous transient studies, we retain the time-dependent term in the momentum equation to describe the transient behavior of the velocity field. Analysis reveals that the steady-state thermal boundary layer develops only after the steady-state velocity field prevails over the entire heat transfer surface. Subsequently, heat diffusion from the wall balances the transverse advection.

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Governing equations and initial and boundary conditions

We consider a flat plate placed in a fluid-saturated porous medium, as shown in Figure 1. Suppose that the fluid is at rest at $t=0$, at which time a sudden uniform and constant pressure gradient dp/dx (<0) is applied along the flat plate to induce a fluid movement. At the same time, the wall temperature of the flat plate is suddenly raised to some high temperature, T_w, above the ambient temperature, T_a, and remains the same thereafter.

Evoking the usual boundary-layer approximations, we can write the governing equations—namely, the unsteady momentum equation based on the Forchheimer extension and the unsteady energy balance equation—in boundary-layer coordinates (t, x, y) as

$$\frac{\rho}{\varepsilon}\frac{du}{dt}=-\frac{dp}{dx}-\frac{\mu}{k}u-\frac{\rho C}{\sqrt{K}}u^2 \tag{1}$$

$$\sigma\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{2}$$

The Darcian (apparent) velocity component in the x direction is denoted by u; T is the local temperature of the fluid, which

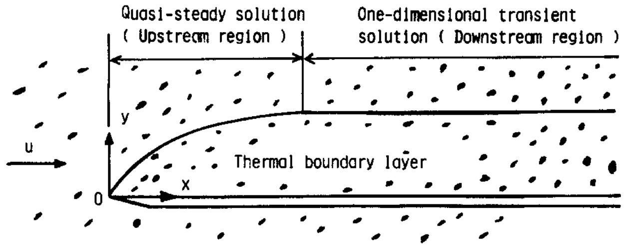

Figure 1 Physical model and its coordinates

is assumed to be in local thermal equilibrium; K is the permeability of the porous medium; ε, its porosity; μ, the fluid viscosity; ρ, its density; α, the effective thermal diffusivity of the saturated porous medium; σ, the ratio of heat capacity of the saturated porous medium to that of the fluid; and C, the empirical constant associated with the porous inertia term. The continuity equation and impermeability at the wall have been utilized already to find the normal velocity component $v=0$, everywhere, and hence $u=u(t)$.

In the analysis, we assume negligible thermal dispersion. Such an assumption holds only when $u_D K^{1/2}/\alpha \ll 10$, according to Cheng.[12] We must satisfy $u_D x/\alpha \gg 1$ for the boundary-layer to hold. Thus we restrict our discussion to values within the range $1 \ll u_D x/\alpha \ll 10x/K^{1/2}$. As $x/K^{1/2}$ is usually very large, $u_D x/\alpha$ may well be within the range, except in the vicinity of the leading edge where the boundary-layer theory itself fails.

The following are the associated initial and boundary conditions for $u(t)$ and $T(t, x, y)$.

- *Initial conditions:*

$$u(0)=0 \tag{3}$$

$$T(0, x, y)=T_a \quad \text{for } x \geqq 0 \text{ and } y \geqq 0 \tag{4}$$

- *Boundary conditions:*

$$T(t, 0, y)=T_a \quad \text{for } t>0 \text{ and } y>0 \tag{5}$$

$$T(t, x, 0)=T_w \quad \text{for } t>0 \text{ and } x \geqq 0 \tag{6}$$

$$T(t, x, \infty)=T_a \quad \text{for } t>0 \text{ and } x \geqq 0 \tag{7}$$

Solving the momentum equation

After some manipulation, an exact solution to the momentum equation subjected to boundary condition Equation 3, may be equation subjected to boundary condition Equation 3, may be obtained as

$$u=u_D \frac{2(1-e^{-(\gamma t/t_D)})}{\gamma+1+(\gamma-1)e^{-(\gamma t/t_D)}} \tag{8}$$

where

$$u_D=\frac{K}{\mu}\left(-\frac{dp}{dx}\right) \tag{9}$$

$$t_D=\frac{\rho K}{\varepsilon \mu} \tag{10}$$

$$\gamma=(1+4\mathrm{Re}^*)^{1/2} \tag{11}$$

$$\mathrm{Re}^*=\frac{c\rho C u_D K^{1/2}}{\mu}=\frac{\rho C K^{3/2}}{\mu^2}\left(-\frac{dp}{dx}\right) \tag{12}$$

is the microscale Reynolds number based on the square root of the permeability and the Darcy flow velocity u_D. For Darcy flow, we may set $C=0$ or $\mathrm{Re}^*=0$; hence $\gamma=1$. Then, Equation 8 reduces to

$$u=u_D(1-e^{-(t/t_D)}) \tag{13}$$

which is identical to the expression we derived earlier[13] for transient film condensation over a vertical flat plate, except that u_D is replaced by the steady falling velocity for the case of film condensation. The unsteady term in the momentum equation plays an important role when $t<t_D$. For example, Vafai and Tien's experimental data on air, water, and engine oils[14] correspond to $K \simeq 10^{-4}$–$10^{-6}\,\mathrm{m}^2$ and $\mu/\rho \simeq 10^{-5}$–$10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, for which we can roughly estimate that $\varepsilon t_D \simeq 0.1$–$10^2\,\mathrm{s}$. Thus there are certainly cases for which the unsteady term in the momentum equation cannot be neglected at the initial period.

Preliminary consideration of the energy equation

Although we have established the transient velocity solution, $u(t)$, the energy equation, Equation 2, is still formidable. Thus we want to find approximate expressions for the temperature field $T(t, x, y)$, by transforming the energy equation under appropriate scales. In order to extract proper scales, we integrate Equation 2 from the wall to infinity:

$$\sigma \frac{\partial}{\partial t}\int_0^\infty (T-T_a)\,dy+u\frac{\partial}{\partial x}\int_0^\infty (T-T_a)\,dy=-\alpha \left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{14}$$

Notation

C	Empirical constant for the Forchheimer term
E	Shape factor associated with the temperature profile
h	Local heat transfer coefficient
K	Permeability
k	Effective thermal conductivity of the saturated porous medium
p	Pressure
Re^*	Microscale Reynolds number representing the inertia effect
T	Temperature
t	Time
u, v	Darcian velocity components
x, y	Boundary layer coordinates

Greek symbols

α	Effective thermal diffusivity of the saturated porous medium
γ	$(1+4\mathrm{Re}^*)^{1/2}$
δ	Vertical length scale for the thermal boundary layer
ε	Porosity of the porous medium
ζ, η	Transformed vertical coordinates
θ	Dimensionless temperature
μ	Fluid viscosity
ξ	Transformed coordinate
ρ	Fluid density
σ	Heat capacity ratio, saturated porous medium to fluid

Subscripts

a	Ambient
D	Darcy flow
w	Wall

We now introduce a vertical length scale δ (proportional to the boundary-layer thickness) for the temperature profile:

$$\frac{T-T_a}{T_w-T_a}=\theta\left(\frac{y}{\delta}\right) \tag{15}$$

Substituting Equation 15 into Equation 14, we obtain the first-order linear hyperbolic equation:

$$\sigma\frac{\partial\delta^2}{\partial t}+u\frac{\partial\delta^2}{\partial x}=E\alpha \tag{16}$$

where

$$E=\frac{-2\left.\dfrac{d\theta(\eta)}{d\eta}\right|_{\eta=0}}{\displaystyle\int_0^\infty \theta(\eta)\,d\eta} \tag{17a}$$

$$\eta=\frac{y}{\delta} \tag{17b}$$

The shape factor, E, may easily be calculated for the given function $\theta(\eta)$. The method of characteristics may be used to find a general solution to Equation 16:

$$\delta^2(t,x)=E\alpha\frac{t}{\sigma}+f\left(x-\frac{1}{\sigma}\int_0^t u\,dt\right) \tag{18}$$

The initial and boundary conditions, namely,

$$\delta^2(0,x)=\delta^2(t,0)=0 \tag{19}$$

provide the function $f(\xi)$:

$$f(\xi)=0 \qquad \text{for } \xi\geqq 0 \tag{20a}$$

$$f(\xi)=-\frac{E\alpha}{\sigma}t^*(\xi) \qquad \text{for } \xi\leqq 0 \tag{20b}$$

where

$$\xi(t^*)=-\frac{1}{\sigma}\int_0^{t^*}u\,dt$$

$$=-\frac{2u_D t_D}{\sigma(\gamma+1)}\left\{\frac{t^*}{t_D}+\frac{2}{\gamma-1}\ln\left[1-\frac{\gamma-1}{2\gamma}\left(1-e^{-(\gamma t^*/t_D)}\right)\right]\right\} \tag{21}$$

The preliminary consideration of the energy equation reveals two distinct vertical length scales:

$$\delta=\left(\frac{E\alpha}{\sigma}t\right)^{1/2} \qquad \text{for } x>\frac{1}{\sigma}\int_0^t u\,dt \tag{22a}$$

$$\delta\simeq\left(E\alpha\frac{x}{\bar{u}(t)}\right)^{1/2} \qquad \text{for } x<\frac{1}{\sigma}\int_0^t u\,dt \tag{22b}$$

where

$$\bar{u}(t^*)=\frac{1}{t^*}\int_0^{t^*}u(t)\,dt \tag{23}$$

We utilize these two length scales to transform the energy equation.

One-dimensional transient solution (downstream asymptotic solution for $\xi>0$)

Adopting the length scale given by Equation 22a, we introduce the following transformed variable appropriate for the downstream region:

$$\eta=\frac{y}{(\alpha t/\sigma)^{1/2}} \tag{24}$$

for

$$\xi=x-\frac{1}{\sigma}\int_0^t u\,dt>0 \tag{25}$$

Note that the downstream region satisfying the above equation diminishes as time elapses. The energy equation, Equation 2, written in (t, x, y) can be transformed into the new coordinates (ξ, η, t) as

$$\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{2}\eta\frac{\partial\theta}{\partial\eta}=t\frac{\partial\theta}{\partial t} \tag{26}$$

for $\xi>0$. Note that the transformed variable ξ does not appear in Equation 26. The term on the right-hand side of Equation 26 may be dropped within a short time after the wall temperature is raised. Then, we have

$$\frac{d^2\theta}{d\eta^2}+\frac{1}{2}\eta\frac{d\theta}{d\eta}=0 \tag{27}$$

which is subjected to the boundary conditions of Equations 6 and 7, namely,

$$\theta(0)=1 \quad \text{and} \quad \theta(\infty)=0 \tag{28a,b}$$

in terms of $\theta(\eta)$. Thus we find

$$\frac{T-T_a}{T_w-T_a}=\operatorname{erfc}\left(\frac{y}{2(\alpha t/\sigma)^{1/2}}\right) \tag{29}$$

where erfc is the complementary error function.

This expression automatically satisfies the initial condition, Equation 4. Equation 29 is identical to the exact solution of transient one-dimensional (1-D) heat conduction in a semi-infinite porous medium subjected to a sudden temperature rise in its bounding heat transfer surface. Thus the effect of advection is not felt in the downstream thermal boundary layer, where $\xi>0$.

Quasi-steady-state solution (upstream asymptotic solution for $\xi<0$)

For the upstream region, we use the vertical length scale given by Equation 22b, namely,

$$\delta\propto(\alpha x/\bar{u})^{1/2}\simeq(\alpha x/u)^{1/2}$$

and introduce the following transformed variable:

$$\zeta=\frac{y}{(\alpha x/u)^{1/2}} \tag{30}$$

for

$$\xi=x-\frac{1}{\sigma}\int_0^t u\,dt<0 \tag{31}$$

The upstream region satisfying the condition of Equation 31 extends farther from the leading edge as time elapses. Next, we transform the energy equation, Equation 2, into the coordinates (ξ, ζ, x) as

$$\frac{\partial^2\theta}{\partial\zeta^2}+\frac{1}{2}\zeta\frac{\partial\theta}{\partial\zeta}=x\frac{\partial\theta}{\partial x}+\frac{1}{2}\left(\frac{\sigma x}{ut}\right)\left(\frac{d\ln u}{d\ln t}\right)\zeta\frac{\partial\theta}{\partial\zeta} \tag{32}$$

For a sufficiently small x, the terms on the right-hand side of

Equation 32 may be dropped, giving

$$\frac{d^2\theta}{d\zeta^2}+\tfrac{1}{2}\zeta\frac{d\theta}{d\zeta}=0 \tag{33}$$

Integration of Equation 33, using the boundary conditions of Equations 6 and 7, again yields a complementary error function:

$$\frac{T-T_a}{T_w-T_a}=\mathrm{erfc}\left(\frac{y}{2(\alpha x/u(t))^{1/2}}\right) \tag{34}$$

Thus for the upstream region extending from the leading edge, the foregoing quasi-steady-state solution holds.

Results and discussion

The transient velocity generated from Equation 8 is shown in Figure 2 for $Re^*=0$ (Darcy flow), 1, and 10. The porous-medium inertia tends to decrease velocity, as expected. The time scale (t_D/γ) diminishes for large Re^*; thus the steady velocity level is reached much faster for highly non-Darcy flows. Although the velocity states can be divided into transient and steady states by the line

$$t=\frac{t_D}{\gamma} \tag{35}$$

the 1-D transient solution and the quasi-steady-state solution for the temperature field should be classified by the limiting characteristic line, $\xi=0$, or

$$\frac{\sigma x}{u_D t_D}=\frac{2}{\gamma+1}\left\{\frac{t}{t_D}+\frac{2}{\gamma-1}\ln\left(1-\frac{\gamma-1}{2\gamma}\left[1-e^{-(\gamma t/t_D)}\right]\right)\right\} \tag{36}$$

Figure 3 shows the regime map constructed for the case of $Re^*=0$ (Darcy flow). Equations 35 and 36 were plotted on the

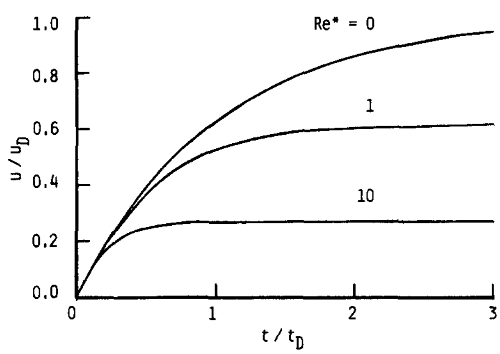

Figure 2 Porous-media inertia effect on transient velocity

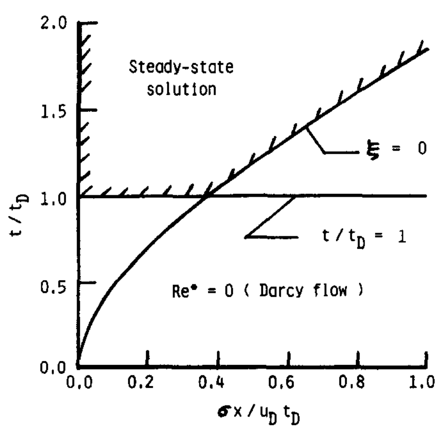

Figure 3 Regime diagram for steady-state and transient solutions

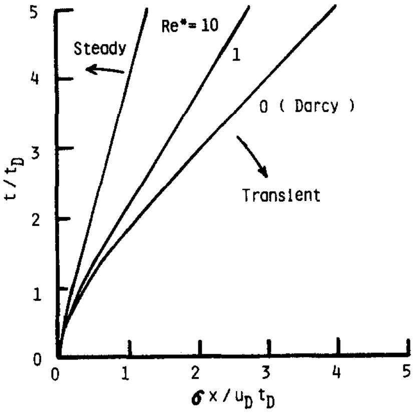

Figure 4 Porous-medium inertia effect on regime diagram

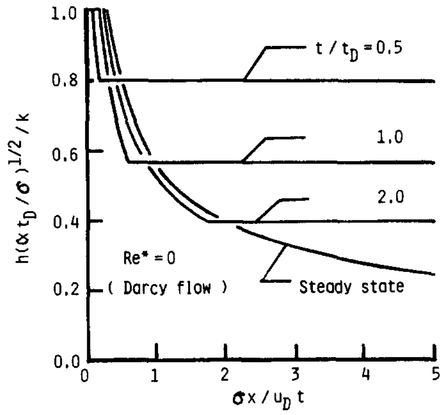

Figure 5 Transient local heat transfer coefficient

$(\sigma x/t_D u_D)-(t/t_D)$ plane. The steady-state thermal boundary layer should be established in the upper left region bounded by

$$\frac{t}{t_D}>\frac{1}{\gamma}\qquad \xi<0 \quad\text{and}\quad x>0 \tag{37}$$

The effect of Re^* on the limiting characteristic line is shown in Figure 4. For the larger Re^*, covering a plate with the steady-state thermal boundary layer takes longer, because porous-medium inertia decreases the velocity.

The local heat transfer coefficient, h, of primary interest can be evaluated from both the 1-D solution, Equation 29, and the quasi-steady-state solution, Equation 34, as

$$\frac{h(\alpha t_D/\sigma)^{1/2}}{k}=-\left(\frac{\alpha t_D}{\sigma}\right)^{1/2}\left.\frac{\partial\theta}{\partial y}\right|_{y=0}$$

$$=\begin{cases}\dfrac{1}{\sqrt{\pi}}\left(\dfrac{t}{t_D}\right)^{-1/2} & \text{for } \xi>0 \quad (38a)\\[2ex] \dfrac{1}{\sqrt{\pi}}\left[\dfrac{2(1-e^{-(\gamma t/t_D)})}{\gamma+1+(\gamma-1)e^{-(\gamma t/t_D)}}\right]^{1/2}\left(\dfrac{\sigma x}{u_D t_D}\right)^{-1/2} & \text{for } \xi<0 \quad (38b)\end{cases}$$

where k is the effective thermal conductivity of the saturated porous medium. The transient behavior of the local heat transfer coefficient is shown in Figure 5 for $Re^*=0$ (Darcy flow). The quasi-steady-state solution, Equation 38b, which prevails in the upstream region (extending from the leading edge), is connected to the 1-D transient solution, Equation 38a (which is a horizontal line). As time elapses, the quasi-steady-state solution region extends further downstream, and the solution curve asymptotically approaches the steady-state solution curve.

Conclusions

Transient non-Darcy forced convection from a flat plate in a fluid-saturated porous medium was investigated by means of a quasi-similarity transformation. The exact solution to the unsteady momentum equation was obtained. The thermal boundary-layer length scales needed for transformation of the unsteady energy equation were extracted by solving the integral energy equation, utilizing the method of characteristics.

The exact velocity solution reveals that the porous-medium inertia decreases velocity and shortens the time required to reach the steady-state velocity. Similarity consideration of the energy equation, however, suggests that the time-dependent temperature field can be divided into downstream and upstream regions. The temperature fields follow the 1-D transient solution and the quasi-steady-state solution, respectively. Subsequently, the two distinct expressions for the local heat transfer coefficient were connected to describe its time-dependent distribution along the plate. As the porous-medium inertia becomes significant, establishment of the steady-state thermal boundary layer is clearly delayed because of the decrease in velocity.

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